New equations for maximal curves

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Workshop on algebraic curves and function fields over a finite fields
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- $K = \overline{F}_q$
Notation

- $\mathbb{K} = \overline{\mathbb{F}}_q$

- $\mathcal{X} \subseteq \mathbb{P}^r(\mathbb{K})$ projective, geometrically irreducible algebraic curve defined over $\mathbb{F}_q$
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- $g = g(\mathcal{X})$ genus of $\mathcal{X}$
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- $g = g(\mathcal{X})$ genus of $\mathcal{X}$

- $\mathcal{X}(\mathbb{F}_q)$ set of $\mathbb{F}_q$-rational places of $\mathcal{X}$

  If $\mathcal{X}$ is non-singular, we can identify places and points of $\mathcal{X}$, and

  $$\mathcal{X}(\mathbb{F}_q) = \mathcal{X} \cap \text{PG}(r, q)$$
Maximal curves

Theorem (Hasse-Weil bound)

Let $\mathcal{X}$ be a curve defined over $\mathbb{F}_q$. Then

$$||\mathcal{X}(\mathbb{F}_q)| - (q + 1)|| \leq 2g\sqrt{q}$$
Maximal curves

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Definition

$X$ is $\mathbb{F}_{q^2}$-maximal if

$$|X(\mathbb{F}_{q^2})| = q^2 + 1 + 2gq$$
Maximal curves

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Definition

\( \mathcal{X} \) is \( \mathbb{F}_{q^2} \)-maximal if \( |\mathcal{X}(\mathbb{F}_{q^2})| = q^2 + 1 + 2gq \)

Example

Hermitian curve:

\[
\mathcal{H}_q : X^q + X = Y^{q+1}, \quad q = p^h
\]

\[
g = q(q - 1)/2 \quad |\mathcal{H}_q(\mathbb{F}_{q^2})| = q^3 + 1
\]
Riemann Hypothesis for curves over finite fields

- Riemann zeta function:

\[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \]
Riemann Hypothesis for curves over finite fields

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- Zeta function of the curve \( X \) defined over \( \mathbb{F}_q \) with genus \( g \):
  \[ \zeta(X, s) = \sum_D N(D)^{-s} \]
  sum over all effective \( \mathbb{F}_q \)-rational divisors \( D \), \( N(D) = q^{\deg(D)} \)
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  \( \zeta(\mathcal{X}, s) \) defines a meromorphic function over \( \mathbb{C} \)
  - Let \( t = q^{-s} \), then
  \[ \zeta(\mathcal{X}, s) = Z(\mathcal{X}, t) = \frac{L(\mathcal{X}, q)(t)}{(1 - t)(1 - qt)} \]
  where \( L(\mathcal{X}, q)(t) \) is the \( L \)-polynomial of \( \mathcal{X} \)
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Theorem (Hasse-Weil)

- \( L(\mathcal{X}, q)(t) \in \mathbb{Z}[t], \ \text{deg} L(\mathcal{X}, q) = 2g \)
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- $L(X, q)(t) \in \mathbb{Z}[t]$, $\deg L(X, q) = 2g$
- $L(X, q)$ has constant term 1 and leading coefficient $q^g$
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If $L(\mathcal{X}, q)(t) = \prod_{i=1}^{2g}(1 - \omega_i t)$, then

- $\omega_i \cdot \omega_i + g = q$ for all $i = 1, \ldots, 2g$
- $|\omega_i| = \sqrt{q}$ for all $i = 1, \ldots, 2g$
- $\Re(s) = \frac{1}{2}$
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  If \( L(\mathcal{X}, q)(t) = \prod_{i=1}^{2g} (1 - \omega_i t) \), then

- \( |\mathcal{X}(\mathbb{F}_{q^m})| = q^m + 1 - (\omega_1^m + \ldots + \omega_{2g}^m) \) for all \( m \geq 1 \)
- up to reordering, \( \omega_i \cdot \omega_{i+g} = q \) for all \( i = 1, \ldots, g \)
Theorem (Hasse-Weil)

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   - If $L(X, q)(t) = \prod_{i=1}^{2g}(1 - \omega_i t)$, then
3. $|X(\mathbb{F}_{q^m})| = q^m + 1 - (\omega_1^m + \ldots + \omega_{2g}^m)$ for all $m \geq 1$
4. up to reordering, $\omega_i \cdot \omega_{i+g} = q$ for all $i = 1, \ldots, g$
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This is the Riemann Hypothesis for curves over finite fields

The Hasse-Weil bound follows as a corollary

For maximal curves over \( \mathbb{F}_q \), \( \omega_i = -\sqrt{q} \) for all \( i = 1, \ldots, 2g \)
Application of maximal curves to AG codes

- $C [n, k, d]_q$-code associated to $X(\mathbb{F}_q)$
Application of maximal curves to AG codes

- $C [n, k, d]_q$-code associated to $\mathcal{X}(\mathbb{F}_q)$
- $\delta = n - k + 1 - d$ Singleton defect
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- $\Delta = \delta / n$ relative Singleton defect
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- By Riemann-Roch Theorem, if $\deg(X) > 2g - 2$, then
  \[ \Delta \leq \frac{g(X)}{|X(\mathbb{F}_q)|} \]
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*Good* curves for AG codes: curves with many rational points with respect to their genus, with explicit equations
Theorem (Serre, 1987)

If $X$ is $\mathbb{F}_q$-maximal and $\phi : X \to Y$ is a non-constant morphism defined over $\mathbb{F}_q$, then $Y$ is $\mathbb{F}_q$-maximal.
Coverings of curves

Theorem (Serre, 1987)

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Theorem

Let $G$ be a finite group of $\mathbb{F}_q$-automorphisms of $X$, that is

$$G < \text{Aut}_{\mathbb{F}_q}(X) = \{ \phi : X \to X \mid \phi \text{ automorphism defined over } \mathbb{F}_q \}$$

Then the quotient curve $X/G$ is $\mathbb{F}_q$-maximal.
\( \mathbb{F}_{q^2} \)-maximal curves not covered by \( \mathcal{H}_q \)

**Theorem (Giulietti-Korchmáros 2009)**

Let \( n \) be a power of a prime \( p \), \( q = n^3 \). The GK-curve

\[
\mathcal{Y}_n : \begin{cases} 
Z \frac{n^3+1}{n+1} = Y^{n^2} - Y \\
X^n + X = Y^{n+1} 
\end{cases}
\]

is \( \mathbb{F}_{q^2} \)-maximal. If \( q > 8 \), \( \mathcal{Y}_n \) is not \( \mathbb{F}_{q^2} \)-covered by \( \mathcal{H}_q \).
\( \mathbb{F}_{q^2} \)-maximal curves not covered by \( \mathcal{H}_q \)

**Theorem (Giulietti-Korchmáros 2009)**

If \( p \) is an \( n \)-th power of a prime, \( q = n^3 \). The GK-curve

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Generalized by Garcia-Gunerí-Stichtenoth:

\( \mathbb{F}_{q^2} \)-maximal curves not Galois-covered by \( \mathcal{H}_q \)

Guralnick-Malmskog-Pries: automorphism group of the GGS-curve
Automorphisms of $\mathcal{Y}_n$

$$|\text{Aut}(\mathcal{Y}_n)| = n^3(n^3 + 1)(n^2 - 1)(n^2 - n + 1) \sim 4g^2$$
Automorphisms of $\mathcal{Y}_n$

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- $\text{Aut}(\mathcal{Y}_n)$ has a subgroup isomorphic to

$$SU(3, n) \times \mathbb{Z}_i,$$

$$i = \begin{cases} 
 n^2 - n + 1, & \text{if } 3 \nmid n + 1 \\
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  \end{cases}$$

- $\text{Aut}(\mathcal{Y}_n)$ has a central subgroup $\Lambda$ of size $n^2 - n + 1$ such that $\text{Aut}(\mathcal{Y}_n)/\Lambda \cong \text{PGU}(3, n) = \text{Aut}(\mathcal{H}_n)$

and the action of $\text{Aut}(\mathcal{Y}_n)/\Lambda$ on the orbits of $\Lambda$ is equivalent to that of $\text{PGU}(3, n)$ on $\mathcal{H}_n$
Quotient curves of $\mathcal{Y}_n$

**Problem**

*Investigate quotient curves of $\mathcal{Y}_n$*
Quotient curves of $\mathcal{V}_n$

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Investigate quotient curves of $\mathcal{V}_n$

- Find automorphism groups of $\mathcal{V}_n$
- Find the genus of quotients of $\mathcal{V}_n$
Quotient curves of $\mathcal{Y}_n$

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- Find explicit equations for quotients of $\mathcal{Y}_n$
- Are the quotients (Galois) covered by the Hermitian curve?
Problem

*Investigate quotient curves of \( \mathcal{Y}_n \)*

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- Are the quotients (Galois) covered by the Hermitian curve?
  
  (Tafazolian - Teherán-Herrera - Torres, Bartoli - Speziali)
Quotient curves of $\mathcal{Y}_n$: examples

\[ n = p^h, \quad b \mid h, \quad c^{n-1} = -1, \quad L \leq \text{Aut}(\mathcal{Y}_n)_{P_\infty}, \quad |L| = p^b \]

\[ \mathcal{Y}_n/L : \begin{cases} cY^{n+1} = \sum_{i=0}^{h/b-1} X^{p^{ib}} \\ Z^{n^2 - n + 1} = Y^{n^2} - Y \end{cases} \]

If $n > p^{2b} + p^b$, then $\mathcal{Y}_n/L$ is not covered by $\mathcal{H}_{n^3}$
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$$s \mid n^2 - n + 1, \quad M = \{(X, Y, Z, T) \mapsto (X, Y, \lambda Z, T) \mid \lambda^s = 1\} \leq \text{Aut}(\mathcal{Y}_n)$$

$$\mathcal{Y}_n / M : \begin{cases} Y^{n+1} = X^n + X \\ Z^{\frac{n^2-n+1}{s}} = Y^{n^2} - Y \end{cases}$$

If $n > s(s + 1)$, then $\mathcal{Y}_n / M$ is not covered by $\mathcal{H}_{n^3}$
An equivalent form of $\mathcal{Y}_n$

Proposition

The GK-curve $\mathcal{Y}_n$ is projectively equivalent over $\mathbb{F}_{n^2}$ to the space curve

$$\mathcal{X} : \begin{cases} Z^{n^2-n+1} = Y \frac{X^{n^2} - X}{X^{n+1} - 1} \\ Y^{n+1} = X^{n+1} - 1 \end{cases}$$
An equivalent form of $\mathcal{Y}_n$

**Proposition**

The GK-curve $\mathcal{Y}_n$ is projectively equivalent over $\mathbb{F}_{n^2}$ to the space curve

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Hence $\text{Aut}(\mathcal{Y}_n) = A \cdot \text{Aut}(\chi) \cdot A^{-1}$, where

$$A = \begin{pmatrix} \rho & 0 & 0 & \rho^n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \text{ with } \rho + \rho^n = 1.$$
A criterion for maximal curves

Proposition

Let \( a, c \in \mathbb{F}_q \), \( b \in \mathbb{F}_{q^2} \) with \( ac - b^{q+1} \neq 0 \). Let

\[
g(X) = aX^{q+1} + (b^q + b)X^q + bX + c
\]

and \( f(X) \in \mathbb{F}_{q^2}[X] \) a divisor of \( g(X) \) with \( \deg(f) \leq d \), where \( d \) is a divisor of \( q + 1 \). If

\[
f(X) \frac{q+1}{d} - 1 - \frac{g(X)}{f(X)}
\]

is the \( d \)-th power of a polynomial \( h(X) \in \mathbb{F}_{q^2}[X] \), then the curve with equations

\[
\begin{cases}
Z^{\frac{q+1}{d}} = Y h(X) \\
Y^d = f(X)
\end{cases}
\]

is \( \mathbb{F}_{q^2} \)-maximal.
A family of Galois subcovers of \( \mathcal{X} \)

\[
G = \left\{ (X, Y, Z, T) \mapsto (aX, bY, \lambda Z, T) \mid a^{n+1} = b^{n+1} = 1, \lambda^{n^2-n+1} = ab \right\}
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A family of Galois subcovers of $\mathcal{X}$

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\[ L = \left\{ (X, Y, Z, T) \mapsto (\lambda^3 b^n X, bY, \lambda Z, T) \mid b^{n+1} = \lambda^{n+1} = 1 \right\} \leq G \]
A family of Galois subcovers of $X$

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$[\mathbb{K}(x, y, z) : Fix(L)] = (n + 1)^2, \quad \mathbb{K}(x^{n+1}, y^{n+1}, z^{n+1}) \subseteq Fix(L),$

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$\Longrightarrow Fix(L) = \mathbb{K}(x^{n+1}, y^{n+1}, z^{n+1})$
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$$\implies \text{Fix}(L) = \mathbb{K}(x^{n+1}, y^{n+1}, z^{n+1})$$

Define $u = x^{\frac{n+1}{d_1}}$, $v = y^{\frac{n+1}{d_2}}$, $w = z^{\frac{n+1}{d_3}}$, with $d_1, d_2, d_3$ divisors of $n+1$
A family of Galois subcovers of $\mathcal{X}$

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$\text{Fix}(L) \subseteq \mathbb{K}(u, v, w) \subseteq \mathbb{K}(x, y, z)$ and $\mathbb{K}(x, y, z)/\text{Fix}(L)$ is Galois
A family of Galois subcovers of $\mathcal{X}$

\[ G = \{(X, Y, Z, T) \mapsto (aX, bY, \lambda Z, T) \mid a^{n+1} = b^{n+1} = 1, \lambda^{n^2-n+1} = ab\} \]

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Then $\mathbb{K}(x, y, z)/\mathbb{K}(u, v, w)$ is Galois, $\mathbb{K}(u, v, w) = \text{Fix}(H)$ with $H \leq L$
A family of Galois subcovers of $\mathcal{X}$, II

$$v^{d_2} = u^{d_1} - 1, \quad w^{d_3(n^2-n+1)} = \alpha,$$

with

$$\alpha = u^{d_1} \left( u^{d_1} - 1 \right) \left( \frac{u^{d_1(n-1)} - 1}{u^{d_1} - 1} \right)^{n+1} \in \mathbb{K}(u, v)$$
A family of Galois subcovers of $\mathcal{X}$, II

\[ v^{d_2} = u^{d_1} - 1, \quad w^{d_3(n^2-n+1)} = \alpha, \]

with \[ \alpha = u^{d_1} \left( u^{d_1} - 1 \right) \left( \frac{u^{d_1(n-1)} - 1}{u^{d_1} - 1} \right)^{n+1} \in \mathbb{K}(u, v) \]

The greatest common divisor between $d_3(n^2 - n + 1)$ and the weights of places of $\mathbb{K}(u, v)$ in $\text{div}(\alpha)$ is

\[ M = \text{gcd} \left( d_1, d_2, d_3(n^2 - n + 1) \right) \]
\[ v^{d_2} = u^{d_1} - 1, \quad w^{d_3(n^2-n+1)} = \alpha, \]

with \[ \alpha = u^{d_1} \left( u^{d_1} - 1 \right) \left( \frac{u^{d_1(n-1)} - 1}{u^{d_1} - 1} \right)^{n+1} \in \mathbb{K}(u, v) \]

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\[ M = \gcd \left( d_1, d_2, d_3(n^2 - n + 1) \right) \]

If \( M = 1 \): irreducible equations of a double Kummer extension
A family of Galois subcovers of \( \mathcal{X} \), II

\[ v^{d_2} = u^{d_1} - 1, \quad w^{d_3(n^2-n+1)} = \alpha, \]

with \[ \alpha = u^{d_1}(u^{d_1} - 1) \left( \frac{u^{d_1(n-1)} - 1}{u^{d_1} - 1} \right)^{n+1} \in \mathbb{K}(u, v) \]

The greatest common divisor between \( d_3(n^2 - n + 1) \) and the weights of places of \( \mathbb{K}(u, v) \) in \( \text{div}(\alpha) \) is

\[ M = \text{gcd} \left( d_1, d_2, d_3(n^2 - n + 1) \right) \]

If \( M = 1 \): irreducible equations of a double Kummer extension

For \( M \geq 1 \): factorize and get irreducible equations

\[ \mathcal{X}/H : \begin{cases} W^{\frac{d_3(n^2-n+1)}{M}} = U^{\frac{d_1}{M}} V^{\frac{d_2}{M}} \left( \frac{U^{d_1(n-1)} - 1}{U^{d_1} - 1} \right)^{\frac{n+1}{M}} \\ V^{d_2} = U^{d_1} - 1 \end{cases} \]
A family of Galois subcovers of $X$: degree

\[
in \mathbb{K}(x, y, z) : \deg(x)_0 = [\mathbb{K}(x, y, z) : \mathbb{K}(x)] = n^3 + 1
\]

hence \[
[\mathbb{K}(x, y, z) : \mathbb{K}(u)] = \deg(x \frac{n+1}{d_1})_0 = \frac{n + 1}{d_1}(n^2 - n + 1)
\]

so \[
[\mathbb{K}(x, y, z) : \mathbb{K}(u, v)] = \frac{(n + 1)^2(n^2 - n + 1)}{d_1d_2}
\]
A family of Galois subcovers of $\mathcal{X}$: degree

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\text{in } \mathbb{K}(x, y, z) : \quad \deg(x)_0 = [\mathbb{K}(x, y, z) : \mathbb{K}(x)] = n^3 + 1
\]

hence \[ [\mathbb{K}(x, y, z) : \mathbb{K}(u)] = \deg(x^{\frac{n+1}{d_1}})_0 = \frac{n + 1}{d_1}(n^2 - n + 1) \]

so \[ [\mathbb{K}(x, y, z) : \mathbb{K}(u, v)] = \frac{(n + 1)^2(n^2 - n + 1)}{d_1 d_2} \]

Therefore \[ |H| = [\mathbb{K}(x, y, z) : \mathbb{K}(u, v, w)] = \frac{[\mathbb{K}(x, y, z) : \mathbb{K}(u, v)]}{[\mathbb{K}(u, v, w) : \mathbb{K}(u, v)]} \]

\[ = \frac{M(n + 1)^2}{d_1 d_2 d_3} \]
By starting from the values $d_1/M, d_2, d_3$, or from $d_1, d_2/M, d_3$, we get other equations for quotient curves:

$$
\begin{cases}
    W^{d_3(n^2-n+1)} = U^{d_1} \left( U^{d_1}(n-1) - 1 \right) \left( \frac{U^{d_1}(n-1)-1}{U^{d_1}-1} \right)^n, \\
    V^{d_2} = U^{d_1} - 1
\end{cases}
$$

$$
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    W^{d_3(n^2-n+1)} = U^{d_1} \left( U^{d_1}(n-1) - 1 \right) \left( \frac{U^{d_1}(n-1)-1}{U^{d_1}-1} \right)^n, \\
    V^{d_2/M} = U^{d_1} - 1
\end{cases}
$$

of degree $\frac{(n+1)^2}{d_1 d_2 d_3}$ and $\frac{M(n+1)^2}{d_1 d_2 d_3}$, respectively.
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\[
\begin{align*}
W^{d_3(n^2-n+1)} &= U^{d_1/M} \left( U^{d_1/(n-1)} - 1 \right) \left( \frac{U^{d_1/(n-1)} - 1}{U^{d_1/M} - 1} \right)^n, \\
V^{d_2} &= U^{d_1} - 1
\end{align*}
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\end{align*}
\]

of degree $\frac{(n+1)^2}{d_1d_2d_3}$ and $\frac{M(n+1)^2}{d_1d_2d_3}$, respectively.

On the function fields of such curves consider the morphism

\[
(u : v : w^{n^2-n+1/e} : 1), \quad \text{with} \quad e \mid n^2 - n + 1.
\]

Then $K(u, v, s)$, where $s = w^{n^2-n+1/e}$, is the function field of new quotients.
A family of Galois subcovers of $\mathcal{X}$: equations

**Theorem**

$d_1, d_2, d_3 \mid n + 1, \ e \mid n^2 - n + 1, \ M = \gcd\left(d_1, d_2, d_3(n^2 - n + 1)\right)$. The following irreducible equations define $\mathbb{F}_{n^6}$-maximal space curves which are Galois subcovers of $\mathcal{X}$:

\[
C_1: \begin{cases} 
S^{d_3} e = U^{d_1} V^{d_2} \left( \frac{U^{d_1(n-1)} - 1}{U^{d_1} - 1} \right)^{n+1} \\
V^{d_2} = U^{d_1} - 1 
\end{cases}
\]

\[
C_2: \begin{cases} 
S^{d_3} e = U^{d_1} \left( \frac{U^{d_1(n-1)} - 1}{U^{d_1} - 1} \right)^n \\
V^{d_2} = U^{d_1} - 1 
\end{cases}
\]

\[
C_3: \begin{cases} 
S^{d_3} e = U^{d_1} \left( \frac{U^{d_1(n-1)} - 1}{U^{d_1} - 1} \right)^n \\
V^{d_2} = U^{d_1} - 1 
\end{cases}
\]
Plane models of $C_1$, $C_2$, and $C_3$ can be easily obtained for some values of $d_1$, $d_2$, and $d_3$ (for example, when one of them is 1)

When $\frac{(n^2-n+1)M(n+1)^2}{ed_1d_2d_3} = 1$, the curve $C_1$ provides a (possibly plane) model of the GK-curve
Plane models of $C_1$, $C_2$, and $C_3$ can be easily obtained for some values of $d_1$, $d_2$, and $d_3$ (for example, when one of them is 1)

When $\frac{(n^2-n+1)M(n+1)^2}{ed_1d_2d_3} = 1$, the curve $C_1$ provides a (possibly plane) model of the GK-curve

The genera of the curves $C_i$ can be computed by applying Kummer theory to the extensions

$$\mathbb{K}(u, v)/\mathbb{K}(u), \quad \mathbb{K}(u, v)/\mathbb{K}(v), \quad \mathbb{K}(u, v, w)/\mathbb{K}(u, v)$$
**Theorem**

Let $e = n^2 - n + 1$. Then the genera of the curves $C_1$, $C_2$, and $C_3$ are

$$g(C_1) = 1 + \frac{1}{2} \left[ d_1 d_2 \frac{d_3(n^2 - n + 1)}{M}(n - 1) - d_2 \left( \frac{d_1}{M}, \frac{d_3(n^2 - n + 1)}{M} \right) - d_1 \left( \frac{d_2}{M}, \frac{d_3(n^2 - n + 1)}{M} \right) + 
- d_1 d_2 (n - 2) \left( \frac{d_3(n^2 - n + 1)}{M}, \frac{n + 1}{M} \right) - \left( (d_1, d_2) \frac{d_3(n^2 - n + 1)}{M}, \frac{d_1 d_2 n(n - 1)}{M} \right) \right]$$

and, for $i = 2, 3$,

$$g(C_i) = 1 + \frac{1}{2} \left[ hkr(n - 1) - k(h, r) - h(k, r) - hk(n - 2)(r, n + 1) - ((h, k)r, hkn(n - 1)) \right],$$

where $r = d_3(n^2 - n + 1)$, $h = \begin{cases} d_1/M & \text{for } C_2 \\ d_1 & \text{for } C_3 \end{cases}$, $k = \begin{cases} d_2/M & \text{for } C_2 \\ d_2 & \text{for } C_3 \end{cases}.$
Explicit description of $H \leq \text{Aut}(\mathcal{X})$, case 1

\[ d_1 \mid d_3, \quad \text{gcd}(d_1, d_2) = 1 \]
Explicit description of $H \leq \text{Aut}(\mathcal{X})$, case 1

\[ d_1 \mid d_3, \quad \gcd(d_1, d_2) = 1 \]

Then $C_i = \mathcal{X}/H$, where

\[ H = \left\{ (X, Y, Z, T) \mapsto (\lambda^3 b^n X, bY, \lambda Z) \mid b^{\frac{n+1}{d_1 d_2}} = \lambda^\frac{n+1}{d_3} = 1 \right\} \]
Explicit description of $H \leq \text{Aut}(\mathcal{X})$, case 1

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The genus $g_H$ of $\mathcal{X}/H$ can be computed also as

\[ g_H = g_{\tilde{H}} + \frac{d_1 d_2 d_3 (n^3 - 2n^2 + (2 - m)n + m - 1)}{2}, \]

where $m = \gcd(3, (n + 1)/d_3)$ and

\[ g_{\tilde{H}} = 1 + \frac{md_1 d_2 d_3}{2(n + 1)} \left( n + 1 - \frac{n + 1}{md_3} - \gcd\left( \frac{n + 1}{d_1 d_2}, \frac{n + 1}{md_3} \right) - \gcd\left( \frac{n + 1}{d_1 d_2}, \frac{2(n + 1)}{md_3} \right) \right) \]

is the genus of the quotient $\mathcal{H}_n/\tilde{H}$, where $\tilde{H}$ is the projection of $H$ on $\text{PGU}(3, n) = \text{Aut}(\mathcal{H}_n)$.
Explicit description of $H \leq \text{Aut}(\mathcal{X})$, case 2

\[ d_1 \mid d_2, \quad \gcd(d_1, d_3(n^2 - n + 1)) = 1 \]

Then $C_i = \mathcal{X}/H$, where

\[ H = \left\{ (X, Y, Z, T) \mapsto (\lambda^3 b^n X, b Y, \lambda Z) \mid b^{\frac{n+1}{d_2}} = \lambda^{\frac{n+1}{d_1 d_3}} = 1 \right\} \]

The genus $g_H$ of $\mathcal{X}/H$ can be computed also as

\[ g_H = g_{\bar{H}} + \frac{d_1 d_2 d_3 (n^3 - 2n^2 + (2 - m)n + m - 1)}{2} \]

where $m = \gcd(3, (n + 1)/d_1 d_3)$ and

\[ g_{\bar{H}} = 1 + \frac{md_1 d_2 d_3}{2(n + 1)} \left( n + 1 - \frac{n + 1}{md_1 d_3} - \gcd\left( \frac{n + 1}{d_2}, \frac{n + 1}{md_1 d_3} \right) - \gcd\left( \frac{n + 1}{d_2}, \frac{2(n + 1)}{md_1 d_3} \right) \right) \]

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Another family of Galois subcovers of \( \mathcal{X} \)

\[
c \mid n + 1, \quad d \mid n^2 - n + 1
\]

\[
K = \left\{ (X, Y, Z, T) \mapsto (b^{-1}X, bY, \lambda Z, T) \mid b^{\frac{n+1}{c}} = \lambda \frac{n^2-n+1}{d} = 1 \right\}
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Another family of Galois subcovers of $\mathcal{X}$

$$c \mid n+1, \quad d \mid n^2 - n + 1$$

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Morphism: $u = x^{\frac{n+1}{c}}, \quad v = xy, \quad w = z^{\frac{n^2-n+1}{d}}$

Then $v^{n+1} = u^{2c} - u^c, \quad w^d = \alpha$, \hspace{1cm} (1)

with $\alpha = v \frac{u^{c(n-1)} - 1}{u^c - 1} \in \mathbb{K}(u, v)$.

$Fix(K) = \mathbb{K}(u, v, w)$, and we find a simple zero of $\alpha$ in $\mathbb{K}(u, v)$, hence equations (1) are irreducible.
Another family of Galois subcovers of $\mathcal{X}$

$$c \mid n + 1, \quad d \mid n^2 - n + 1$$

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Morphism: $u = x^{n+1}c$, $v = xy$, $w = z^{n^2-n+1}d$

Then $v^{n+1} = u^{2c} - u^c$, $w^d = \alpha,$ \hspace{1cm} (1)

with $\alpha = v \frac{u^{c(n-1)} - 1}{u^c - 1} \in \mathbb{K}(u, v)$.  

$\text{Fix}(K) = \mathbb{K}(u, v, w)$, and we find a simple zero of $\alpha$ in $\mathbb{K}(u, v)$, hence equations (1) are irreducible.

Moreover, the genus of $\mathcal{X}/K$ can be computed by the Riemann-Hurwitz formula on the covering $\mathcal{X} \to \mathcal{X}/K$. 
Another family of Galois subcovers of $\mathcal{X}$

**Theorem**

The Galois subcover $\mathcal{X}/K$ of $\mathcal{X}$ has degree

$$|K| = \frac{n^3 + 1}{cd},$$

irreducible equations

$$\mathcal{X}/K : \begin{cases} W^d = V \frac{U^c(n-1)-1}{U^c - 1} \\ V^{n+1} = U^{2c} - U^c \end{cases},$$

and genus

$$g(\mathcal{X}/K) = 1 + \frac{c}{2} \left[ (d - 1)n^2 + n - d - \gcd(2, (n + 1)/c) \right].$$
New equations of $\mathbb{F}_{n^6}$-maximal curves: the case $n = 5$

For $n = 5$, the previous results provide new equations for the following genera of $\mathbb{F}_{5^6}$-maximal curves:

$$37, 74, 109, 121, 148, 220, 242, 361, 442, 484, 724, 1450,$$
$$160, 233, 469, 478, 496, 737, 1477, 1486.$$
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$$37, 74, 109, 121, 148, 220, 242, 361, 442, 484, 724, 1450,$$

$$160, 233, 469, 478, 496, 737, 1477, 1486.$$

Up to our knowledge, the integers in the second row are new values in the spectrum of genera of $\mathbb{F}_{5^6}$-maximal curves.
Examples of $\mathbb{F}_n^6$-maximal curves not covered by $\mathcal{H}_n^3$

If $\mathcal{Y}$ is $\mathbb{F}_{q^2}$-covered by $\mathcal{H}_q$ with the morphism $\varphi : \mathcal{H}_q \to \mathcal{Y}$, then

\[
L_{\mathcal{H}_q, \mathcal{Y}} := \frac{|\mathbb{F}_{q^2}(\mathcal{H}_q)|}{|\mathbb{F}_{q^2}(\mathcal{Y})|} \leq \deg(\varphi) \leq \frac{2g(\mathcal{H}_q) - 2}{2g(\mathcal{Y}) - 2} =: U_{\mathcal{H}_q, \mathcal{Y}}
\]

If $\left\lceil L_{\mathcal{H}_q, \mathcal{Y}} \right\rceil > \left\lfloor U_{\mathcal{H}_q, \mathcal{Y}} \right\rfloor$, such covering cannot exist.
Examples of $\mathbb{F}_{n^6}$-maximal curves not covered by $\mathcal{H}_{n^3}$

If $\mathcal{Y}$ is $\mathbb{F}_{q^2}$-covered by $\mathcal{H}_q$ with the morphism $\varphi : \mathcal{H}_q \to \mathcal{Y}$, then

$$L_{\mathcal{H}_q, \mathcal{Y}} := \frac{|\mathbb{F}_{q^2}(\mathcal{H}_q)|}{|\mathbb{F}_{q^2}(\mathcal{Y})|} \leq \deg(\varphi) \leq \frac{2g(\mathcal{H}_q) - 2}{2g(\mathcal{Y}) - 2} =: U_{\mathcal{H}_q, \mathcal{Y}}$$

If $\left[ L_{\mathcal{H}_q, \mathcal{Y}} \right] > \left[ U_{\mathcal{H}_q, \mathcal{Y}} \right]$, such covering cannot exist

**Table:** New $\mathbb{F}_{n^6}$-maximal curves not covered by $\mathcal{H}_{n^3}$

<table>
<thead>
<tr>
<th>$g$</th>
<th>$n$</th>
<th>$(d_1, d_2, d_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>233416</td>
<td>17</td>
<td>(1,18,6), (2,9,6), (2,18,3), (2,18,6), (3,18,6), (6,9,6), (6,18,3), (6,18,6), (9,2,6), (9,6,6), (9,18,2), (9,18,6), (18,1,6), (18,2,3), (18,2,6), (18,3,6), (18,6,3), (18,6,6), (18,9,2), (18,9,6)</td>
</tr>
<tr>
<td>233398</td>
<td>17</td>
<td>(9,18,2)</td>
</tr>
<tr>
<td>1064701</td>
<td>23</td>
<td>(1,24,8), (8,3,8), (24,8,1), (24,1,8), (2,24,8), (3,8,8), (3,24,8), (4,24,8), (6,8,8), (6,24,8), (8,3,8), (8,6,8), (8,12,8), (8,24,1), (8,24,2)</td>
</tr>
<tr>
<td>1064689</td>
<td>23</td>
<td>(2,24,8), (4,24,8), (6,8,8), (6,24,8), (8,6,8), (8,12,8)</td>
</tr>
<tr>
<td>3206257</td>
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</tr>
<tr>
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<td>29</td>
<td>(30,10,1), (10,30,1), (10,15,2), (30,2,5), (10,6,5), (10,3,10)</td>
</tr>
<tr>
<td>5570731</td>
<td>32</td>
<td>(33,11,1), (11,33,1), (11,3,11)</td>
</tr>
</tbody>
</table>
A criterion for the degree of a covering

**Remark**

Let $\varphi : \mathcal{H}_q \to \mathcal{Y}$ be an $\mathbb{F}_{q^2}$-covering. If $g(\mathcal{Y}) > f(q)$, where

$$f(q) = \frac{\sqrt{q^5 + 2q^4 + q^3 + q^2 + 2q + 1} - q^2 - 1}{2q},$$

then $\deg(\varphi)$ is uniquely determined by

$$L_{\mathcal{H}_q, \mathcal{Y}} \leq \deg(\varphi) \leq U_{\mathcal{H}_q, \mathcal{Y}}.$$

In fact, the condition $g(\mathcal{Y}) > f(q)$ is equivalent to $U_{\mathcal{H}_q, \mathcal{Y}} - L_{\mathcal{H}_q, \mathcal{Y}} < 1$. 
\( F_{n^6} \)-maximal curves not Galois-covered by \( \mathcal{H}_{n^3} \)

**Proposition**

Let \( n \geq 7 \), \( k \mid n + 1 \) with \( k < \sqrt{n + 1} + 1 \). Define \( d_1 = \frac{n + 1}{k} \), \( d_2 = 1 \), and \( d_3 = n + 1 \). Then the curve

\[
\mathcal{X}/\mathcal{H} : \quad \mathcal{W}^{n^3 + 1} = U^{\frac{n+1}{k}} \left( \frac{U^{n+1}}{k} - 1 \right) \left( \frac{U^{(n^2-1)/k} - 1}{U^{(n+1)/k} - 1} \right)^{n+1}
\]

is not Galois-covered by \( \mathcal{H}_{n^3} \).
Sketch of the proof:
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- If $\mathcal{H}_{n^3} \to \mathcal{X}/H$ is an $\mathbb{F}_{n^6}$-covering, then it has degree $kn$. 
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- If $\mathcal{H}_{n^3} \to \mathcal{X}/H$ is an $\mathbb{F}_{n^6}$-covering, then it has degree $kn$.

- Suppose $\mathcal{X}/H \cong \mathcal{H}_{n^3}/G$ with $G \leq \text{Aut}(\mathcal{H}_{n^3})$, $|G| = kn$. Since $|G| < n^2$, then $G$ fixes a point of $\mathcal{H}_{n^3}(\mathbb{F}_{n^6})$;
\( \mathbb{F}_{n^6} \)-maximal curves not Galois-covered by \( \mathcal{H}_{n^3} \)

Sketch of the proof:

- If \( \mathcal{H}_{n^3} \to \mathcal{X}/H \) is an \( \mathbb{F}_{n^6} \)-covering, then it has degree \( kn \).

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  Since \( |G| < n^2 \), then \( G \) fixes a point of \( \mathcal{H}_{n^3}(\mathbb{F}_{n^6}) \);

- A result of Garcia-Stichtenoth-Xing provides the genus \( g(\mathcal{H}_{n^3}/G) \), depending on certain numerical parameters.
  On the other hand, \( g(\mathcal{X}/H) \) is given by the formulas above.
\( \mathbb{F}_n^6 \)-maximal curves not Galois-covered by \( \mathcal{H}_{n^3} \)

Sketch of the proof:

- If \( \mathcal{H}_{n^3} \rightarrow \mathcal{X}/H \) is an \( \mathbb{F}_n^6 \)-covering, then it has degree \( kn \).

- Suppose \( \mathcal{X}/H \cong \mathcal{H}_{n^3}/G \) with \( G \leq \text{Aut}(\mathcal{H}_{n^3}) \), \( |G| = kn \).
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- A result of Garcia-Stichtenoth-Xing provides the genus \( g(\mathcal{H}_{n^3}/G) \), depending on certain numerical parameters.
  On the other hand, \( g(\mathcal{X}/H) \) is given by the formulas above.

- The two values cannot coincide.
Proposition

Let $n > 7$, $k | n + 1$ with $k < \sqrt{n + 1} + 1$ and $3 \nmid (n + 1)/k$; if $3 | n + 1$, assume also $n > 23$.

Define $d_1 = (n + 1)/k$, $d_2 = n + 1$, and $d_3 = 1$.

Then the curve

$$
\mathcal{X}/\mathcal{H} : \begin{cases} 
W^{n^2-n+1+1} = U^{n+1} \left( U^{\frac{n+1}{k}} - 1 \right) \left( \frac{U^{(n^2-1)/k-1}}{U^{(n+1)/k-1}} \right)^{n+1} \\
V^{n+1} = U^{\frac{n+1}{k}} - 1
\end{cases}
$$

is not Galois-covered by $\mathcal{H}_{n^3}$.
Proposition

\( n \) prime power, \( \gamma \mid n + 1, \delta \mid n^2 - n + 1, \)

define \( c = (n + 1)/\gamma \) and \( d = (n^2 - n + 1)/\delta. \)

Suppose that one of the following holds:

- \( n = 5, \gamma = 2, \) and \( \delta = 1; \)
- \( n \geq 7, \gamma \leq 2, \) and \( \delta \leq (\sqrt{2\gamma n + 1} - 1)/2; \)
- \( n \geq 7, \gamma > 2, \) and \( \gamma\delta(\gamma\delta - \delta - 1) < n. \)

Then the curve

\[
\begin{aligned}
W & \frac{n^2-n+1}{\delta} = V \frac{U(n^2-1)/\gamma - 1}{U(n+1)/\gamma - 1} \\
V^{n+1} = U^{2(n+1)/\gamma} - U^{(n+1)/\gamma}
\end{aligned}
\]

is not Galois-covered by \( \mathcal{H}_{n^3}. \)
Thank you for your attention